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SOLAR CORE HOMOLOGY, SOLAR NEUTRINOS AND HELIOSEISMOLOGY[*]

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ABSTRACT

Precise numerical standard solar models (SSMs) now agree with one another and with helioseismological observations in the convective and outer radiative zones. Nevertheless these models obscure how luminosity, neutrino production and g-mode core helioseismology depend on such inputs as opacity and nuclear cross sections. Although the Sun is not homologous, its inner core by itself is chemically evolved and almost homologous, because of its compactness, radiative energy transport, and ppI-dominated luminosity production. We apply luminosity-fixed homology transformations to the core to estimate theoretical uncertainties in the SSM and to obtain a broad class of non-SSMs, parametrized by central temperature and density and purely radiative energy transport in the core.

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1. Homology and the Solar Core

After more than three decades of nuclear cross section measurements, opacity calculations, and detailed computer evolutionary calculations, standard solar models (SSMs) with the same inputs now agree in their neutrino flux predictions to within about 1%. The theoretical models are now also consistent with precise p-mode helioseismological observations of the Sun's outer radiative zone $x \equiv r/R_\odot = 0.26 - 0.71$ and convective zone $x > 0.71$. If the g-mode helioseismological oscillations have sufficient amplitude, their observation can be expected soon to calibrate the solar inner core, where the thermonuclear luminosity and neutrino production take place. While necessary in the complex convective zone and justified by the precise helioseismological observations, the complexity and numerical form of precise SSMs obscure the simplicity of the solar core and the determinants of solar neutrino fluxes. In order to understand standard and non-standard solar models, we return to the homology methods of Schwarzschild (1958) and Iben (1969 and 1991) used before the advent of fast computers, but with three new features.

Castellani *et al.* (1993) have found that changing input parameters by factors as large as two leads to only homologous changes over 60% by mass of the Sun. In this paper, we explain this remarkable homology and demonstrate that, while the entire Sun is certainly not homologous, the core is homologous enough to be parametrized by its central temperature, T_c and density ρ_c. This (T_c, ρ_c)-parametrization subsumes all astrophysical effects of opacity, composition, and the *ppI* nuclear cross section factors S_{11} and S_{33} (the pp and $^3He - ^3He$ reactions) into the two parameters (T_c, ρ_c), one representation of the central boundary conditions of solar structure. Indeed, any standard or nonstandard solar model that depends principally on radiative energy transport can be parametrized by (T_c, ρ_c) and the remaining nuclear cross section factors S_{34} and S_{17} (the $^3He - ^4He$ and $p - ^7Be$ reactions) (Degli'Innocenti 1994). Even solar models with a non-standard low opacity or low metallicity Z are all essentially parametrized by (T_c, ρ_c) or by S_{11}, the principal cross section factor determining T_c. (See Figure 2 of Hata *et al.* (1994) or Figure 2 of Hata and Langacker (1995); see also Hata (1994).) As the ρ_c dependence of the neutrino fluxes is weak (see section 4 below), Hata and Langacker (1994) were able to show that the 0.7% theoretical uncertainty in T_c, together with the remaining nuclear cross section uncertainties, provide the same theoretical neutrino flux and rate uncertainties and correlations that Bahcall and Ulrich obtained from 1000 Monte Carlo SSM simulations. (See Figures 2-4 and 6-8 of Hata and Langacker (1995).)

The (T_c, ρ_c) parametrization allows analytic estimation of the logarithmic derivatives $\beta_j(i) \equiv \partial \ln \phi(i)/\partial \ln S_j$ of the principal neutrino fluxes $\phi(i)$ with respect to input parameters S_j, which Bahcall and Ulrich (1988) obtained from 1000 Monte Carlo SSMs calculated with *small* changes in input parameters. Because homology makes these logarithmic derivatives constants, from any precise SSM, we can not only estimate theoretical uncertainties, as did Bahcall and Ulrich, but we can now also extrapolate to non-SSMs, so long as the energy transport is primarily radiative.

Our application of homology differs from earlier ones in three ways: (1) We apply homology only to the solar inner core, not to the whole Sun; (2) we do not assume $\rho \sim T^3$ or any polytropic relation; and (3) we use homology to derive the dependence of *core* temperature and density on opacity, nuclear energy generation, and mean molecular weight, at *fixed* luminosity, instead of the dependence of *effective* temperature on luminosity, with fixed opacity and nuclear energy generation (Cox and Giuli 1968).

After accounting for the different energies released when pp, ^7Be, ^8B, and CNO neutrinos are produced, the known solar luminosity $L_\odot$ fixes the total photon energy production

and constrains the neutrino fluxes through the nuclear reactions:

$$\phi(pp) + (0.967)\,\phi(Be) + (0.743)\,\phi(B) + (0.946)\,\phi(CNO) = 6.48 \times 10^{10}\,\mathrm{cm}^{-2}\mathrm{s}^{-1} \quad , \quad (1)$$

for the four principal neutrino fluxes. (See Appendix; Castellani *et al.* (1993); Bludman *et al.* (1993); Hata and Langacker (1994).) For the SSM, the three pp branches I, II, III terminate in the ratio $83.7\% : 16.3\% : 0.02\%$. Differentiating this constraint and assuming these termination ratios continue to hold approximately, we have the constraint

$$\beta_i(pp) + (0.079)\beta_i(Be) + (0.000071)\beta_i(B) + (0.0145)\beta_i(CNO) = 0 \qquad (2)$$

on the logarithmic derivatives of the principal neutrino fluxes with respect to any input parameter S_j. The logarithmic derivatives obtained by Bahcall and Ulrich from their 1000 Monte Carlo SSMs satisfy this sum rule (Bahcall and Ulrich 1988; Turck-Chièze, 1988; Bludman *et al.* 1993; Hata 1994; Hata *et al.* 1994; Hata and Langacker 1994, 1995). For models that depart greatly from the SSM, the three termination ratios in general change significantly from the values used here; in that case, the new ratios can be used as the new constraint, for small deviations from the new SM. The numerical coefficients of equations (1) and (2) also change.

The central temperature and density are outputs characterizing solar models. Along with the nuclear cross sections and chemical composition, given by the vector $\mathbf{X}$ of element abundances by mass, they determine the neutrino fluxes. The logarithmic derivatives of these fluxes with respect to central temperature, $\alpha(i) \equiv \partial \ln \phi(i)/\partial \ln T_c$, satisfy

$$\alpha(pp) + (0.079)\alpha(Be) + (0.000071)\alpha(B) + (0.0145)\alpha(CNO) = 0 \quad , \qquad (3)$$

and are functionals of temperature, density, and composition, that can be approximated by power laws in temperature, if the weak ρ_c dependence is ignored. If $\alpha(pp) = -1.2$, then $\alpha(Be) = 8 \pm 2$, $\alpha(CNO) = 34 \pm 9$, is consistent with this constraint. Once the $ppII$, $ppIII$, and CNO cycles are included, the luminosity constraint prevents the solar core from being strictly homologous and induces a small ρ_c dependence.

2. The Present Sun

The luminosity- and neutrino-generating core and the outer radiative and convective zones are almost decoupled dynamically. For this reason, the solar model inputs of luminosity $L_\odot$ and radius $R_\odot$ almost separately determine the two outputs, the helium abundance Y and the convective zone mixing length (Bahcall and Ulrich 1988; Turck-Chièze *et al.* 1988). This permits us to ignore the convective zone, for which accurate opacities and detailed numerical models are needed, and to concentrate on the outer radiative zone and the inner core. The radius $R_\odot$ being irrelevant to the inner core leaves only the luminosity $L_\odot$ and the mass $M_\odot$ as fundamental parameters.

The equation of hydrostatic equilibrium is

$$-dP/\rho dr = g \equiv Gm/r^2 \qquad (4a)$$

or

$$\equiv 1/\lambda_P = g/(P/\rho) \quad , \qquad (4b)$$

where $\lambda_P \equiv (-d\ln P/dr)^{-1}$ is the pressure scale height. We define the stiffness (effective polytropic exponent) $\Gamma \equiv d\ln P/d\ln \rho \equiv 1 + (1/n_{\mathrm{eff}})$ and effective polytropic index $n_{\mathrm{eff}} \equiv$

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$d\ln\rho/d\ln(P/\rho)$ and write the equation of state as $P/\rho = \Re(T/\mu)(1 + D)$, where $\Re$ is the gas constant, μ is the mean molecular weight and D includes all corrections to the ideal gas equation of state. Then $1 - \Gamma^{-1} \equiv d\ln(P/\rho)/d\ln P \equiv \nabla - \nabla_\mu$, where the thermal gradient $\nabla \equiv d\ln T/d\ln P$ and chemical gradient $\nabla_\mu \equiv d\ln(\mu/(1+D))/d\ln P$. If n_{eff} and $\mu/(1+D)$ were constant, the Sun would be a polytrope of index n_{eff} and thermal gradient $\nabla = 1/(n_{\text{eff}} + 1)$. These conditions obtain in the convective and outer radiative zones of the Sun, but not in the chemically evolved and inhomogeneous inner core. The core structure is usually obtained by evolutionary models that depend on the initial relative metallicity Z/X for the proto-Sun and the present age.

Figures 1 and 2 show the gradients ∇, $1 - \Gamma^{-1}$ and Γ as function of the dimensionless radius $x \equiv r/R_\odot$ and included mass $m/M_\odot$, derived from the SSM of Dearborn 1994. (This model provides dense enough output to allow us to plot the P, ρ, T logarithmic derivatives and agrees well with other SSMs, such as the SSM of Bahcall and Pinsonneault (1992) with helium diffusion.) The Sun's convective zone is an $n_{\text{eff}} = 3/2$ polytrope; the outer radiative zone, approximately an $n_{\text{eff}} = 4.3$ polytrope. The core, chemically evolved and inhomogeneous, contains the Sun's luminosity production and the majority of its mass. This complex structure prevents the entire Sun from being homologous, even though luminosity production and opacity are approximately power laws in individual zones.

The simplicity of the passive outer radiative zone governs any matter-amplified neutrino oscillations which may take place there. The concentration of mass and luminosity production interior to this zone makes it approximately an $n_{\text{eff}} = 4.3$ polytrope, with density scale height $\lambda_\rho = \Gamma\lambda_P$ very nearly constant at $R_\odot/10.5$. With this scale height, if neutrinos of energy E and mass squared difference Δm^2 oscillate with vacuum mixing angle $\sin^2\theta$, the adiabaticity is

$$\mathcal{A} = 5.3 \times 10^7 (\Delta m^2(eV^2)/E(MeV))(\sin^2 2\theta/\cos 2\theta) \quad , \tag{5}$$

and the jump probability $P_j = \exp(-E_{NA}/E)$, with

$$E_{NA} \equiv (\pi^2/2)\mathcal{A}E = 3 \times 10^8 \, \Delta m^2(\sin^2 2\theta/\cos 2\theta) \quad . \tag{6}$$

The constant density scale height of the $n = 4.3$ polytropic outer radiative zone is the principal property of the Sun effecting the Mikheyev-Smirnov-Wolfenstein (MSW) neutrino oscillations. The best small-angle MSW solution to the solar neutrino observations gives $E_{NA} \approx 13$ MeV, so that ^8B neutrinos of energy E=10 MeV oscillate non-adiabatically with $\mathcal{A} = 0.3$ and pp neutrinos of energy E=0.3 MeV oscillate adiabatically with $\mathcal{A} = 10$.

The solar inner core is reasonably inferred to have been initially convective and therefore chemically homogeneous. The thermal gradient was thus initially adiabatic, $\nabla = \nabla_{\text{ad}}$, decreasing since. Meanwhile, the composition gradient ∇_μ has been increasing from zero. At the present epoch, these two evolutionary changes nearly compensate, $\nabla \approx \nabla_\mu$, making $\Gamma \approx 1$, so that $T/\mu \approx$ constant (Figure 2) and the core is quite condensed. Together with the thermal gradient $\nabla \approx \nabla_\mu \approx 1/3$, this makes T^3/ρ constant to within 7% for $x < 0.3$, $m/M_\odot < 0.613$ (Bahcall and Ulrich 1988). This accident of the present epoch makes $(\mu/(1 + D))P \sim \rho^{4/3}$, which, in the chemically *inhomogeneous* core, is *not* an $n_{\text{eff}} = 3$ polytrope. Since $P/\rho \sim T/\mu \simeq$ constant in the core, the core structure is nearly an "isothermal" $n_{\text{eff}} = \infty$ polytrope with T/μ rather than T nearly constant. We note for completeness that a long-lived convective core (such as a present-epoch fully mixed core)

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would not follow the simple homology of this paper, as the simple chemical stratification $T/\mu \simeq$ constant would be violated.

Although far from polytropic, the solar inner core is almost homologous, because over the narrow range of density and temperature in the compact core, the nuclear energy generation and Rosseland mean opacity are approximated by the power laws,

$$\varepsilon = \varepsilon_o \rho^\lambda T^\nu, \qquad \kappa = \kappa_o \rho^n T^{-s} \quad , \tag{7}$$

where $\varepsilon_o, \kappa_o, \mu$ depend on the chemical composition. Outside of the core ranges of density and temperature, these homologous forms would still be valid, but with different exponents. (Alternative forms of energy transport, such as WIMPs, would destroy the homology altogether.) The core, which we define by radius $x < 0.26$ so as to contain 99% of the energy generation and 50% of the mass, has $T = (7.7-15.7) \times 10^6 \, K$, $\rho = (19-154) \, \mathrm{g \, cm^{-3}}$, and a small pressure $P = P_c/12$ at its edge. In this range, we adopt $\lambda = 1$, $\nu \approx 4.24$ for the luminosity generation (dominantly but not exclusively ppI cycle) and fit the inner core OPAL opacity (Iglesias and Rogers 1991; Rogers 1992) by $n \approx 0.43$, $s \approx 2.47$. Our inner core OPAL opacity fit is close to the values $n = 0.5$, $s = 2.5$ illustrated by Figure 10.4 of Böhm-Vitense (1992). For comparison, we also consider the Kramers opacity (Cox and Giuli 1968).

3. Homology Applied to the Solar Core Only

Homology is usually applied to zero-age main sequence (ZAMS), chemically homogeneous stars to derive power-law $L - M$, $R - M$, $L - T_{\mathrm{eff}}$ relations among luminosity, radius and *effective* temperature for *different* stars with the *same* power-law opacity and luminosity generation. As shown in Figure 3, taken from Kippenhahn and Weigert (1990), these homology relations are obeyed by models and actual stars on the ZAMS upper and lower main sequence, which respectively have convective cores or envelopes. For these stars, $L \sim M^{3.35}$, $R \sim M^{0.57}$ and $L \sim M^{3.2}$, $R \sim M^{0.8}$, respectively.

The Sun, however, is not chemically homogeneous, is mostly radiative, and is transitional between pp and CNO burning. Even on the ZAMS, the Sun was not homologous. Figure 3 shows that, for $(0.3 - 1.3)M_\odot$ stars near the Sun, the exponents change rapidly with mass, with $L \sim M^{4.6}$, $R \sim M^{1.4}$ at the solar mass. Therefore, we cannot apply homology to the entire present or ZAMS Sun.

On the other hand, because the core of the Sun has approximately power-law opacity and luminosity generation and has a low pressure boundary condition, $P \approx P_c/12$, we can apply homology to the isolated solar *core*. We do so in a different way: instead of deriving luminosity, mass and radius relations for a *family* of stars having the same opacity and luminosity generation, we derive the dependence of central temperature $T_c(\kappa, \varepsilon, \mu)$ and density $\rho_c(\kappa, \varepsilon, \mu)$ on overall opacity, energy generation, and mean molecular weight, for *one* star of *fixed* luminosity. This enables us to scale from any given SSM to models of the same luminosity with different input parameters.

Assuming an ideal gas equation of state, the mass conservation and hydrostatic equilibrium equations scale as

$$\rho \sim m/r^3 \quad , \quad P \sim m^2/r^4 \quad , \tag{8}$$

where m the mass included inside radius r, so that

$$m/r \sim P/\rho \sim T/\mu \quad , \quad P_\gamma/P \sim T^4/P \sim (\mu^2 m)^2 \quad , \quad T^3/\rho \sim \mu^3 m^2 \quad , \tag{9}$$

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where P and P_γ are the total and radiation pressures. The first and second relations give the virial theorem $m/r \sim T/\mu$. Any given correction D to the ideal gas equation of state can be included in the homology formulas by replacing μ with $\mu/(1+D)$. This substitution ignores any ρ, P, or T dependence in D. Typical corrections include (Bahcall and Ulrich 1988; Bahcall and Pinsonneault 1992): the Debye-Hückel screening effect, contributing $D \simeq -0.014$; photon pressure, contributing $D \simeq +0.001$; quantum degeneracy, contributing a negligibly small effect; and a hypothetical core magnetic field, contributing $D \simeq +0.002$ for a field strength of 10^8 gauss and scaling as the square of the field strength, if the reverse influence of the thermomechanical structure upon the magnetic field is ignored.

The equations for radiative energy transport and thermal steady state are, using equation (4b),

$$\kappa(\ell/m) = 4\pi cG(dP_\gamma/dP) \quad , \quad \varepsilon = d\ell/dm \quad , \tag{10}$$

where κ, ℓ, ε are the Rosseland mean opacity, luminosity, specific thermal energy generation at radius r, so that

$$\ell \sim \mu^4 m^3/\kappa \quad . \tag{11}$$

From equations (9) and (10), we deduce how ℓ scales with μ, m, r:

$$\varepsilon\kappa \sim T^4/P \sim (\mu^2 m)^2 \quad . \tag{12}$$

The quantity $r^{\nu+3\lambda-s+3n} \sim \varepsilon_o\kappa_o\mu^{\nu-s-4}m^{\nu+\lambda-s+n-2}$ can be eliminated to obtain

$$\ell(\mu,m) \sim \varepsilon_o^{-\alpha}\kappa_o^{-\beta}\mu^\gamma m^\delta \quad , \quad \ell(\mu,T) \sim \varepsilon_o^\epsilon\kappa_o^\zeta\mu^{-\eta}T^\theta \quad , \tag{13}$$

where

$$\alpha = \frac{s-3n}{\nu+3\lambda-s+3n}$$
$$\beta = 1+\alpha$$
$$\gamma = 4+3n+\frac{(4+3\lambda+3n)(s-3n)}{\nu+3\lambda-s+3n} \tag{14a}$$
$$\delta = 3+2n+\frac{(2+2\lambda+2n)(s-3n)}{\nu+3\lambda-s+3n} \quad ,$$

$$\epsilon = \frac{3+2n}{2+2\lambda+2n}$$
$$\zeta = 1-\epsilon$$
$$\eta = \frac{(3+2n)(4+3\lambda+3n)}{2+2\lambda+2n}-4-3n \tag{14b}$$
$$\theta = \frac{(3+2n)(\nu+3\lambda-s+3n)}{2+2\lambda+2n}+s-3n \quad .$$

This homology rests on equating the luminosity produced with the luminosity transported in the steady state (10). The exponents in $\ell(\mu,T)$ are different from the exponents in $L(\mu,T_{\text{eff}})$ obtained when homology is applied to an entire star (Cox and Giuli 1968). We do not consider the dependence on surface photon temperature, T_{eff}, but on central temperature T_c.

For $\lambda = 1$, $\nu = 4.24$ and the core opacity laws we consider, some numerical values are given in Table 1. (The table also contains the exponents for the Böhm-Vitense opacity,

with $\nu = 4$ for the *ppI* cycle.) The exponents are insensitive to the temperature exponent $\nu = 4.24$ in the luminosity generation law, but are sensitive to the opacity law. For the OPAL opacity, we obtain $\ell \sim m^{4.81}$, in good agreement with the value $L \sim M^{4.6}$ for the ZAMS Sun in Figure 3.

We are interested in how the temperature varies as function of luminosity generation, opacity and mean molecular weight, for fixed luminosity $L_\odot$.

$$T_c \sim (\mu^\eta / \varepsilon_o^\epsilon \kappa_o^\zeta)^{1/\theta} \quad . \tag{15a}$$

For the OPAL opacity function, we obtain the differential relation:

$$d\ln T_c = (0.215)d\ln\mu - (0.133)d\ln\varepsilon_o - (0.0344)d\ln\kappa_o + (0.167)d\ln L_\odot \quad , \tag{15b}$$

showing how the central temperature in any compact radiative core must change with input parameters. The central temperature is most sensitive to the chemically evolved mean molecular weight and to the overall luminosity generation ε_o, and much less sensitive to the opacity, κ_o. This is expected, since the core structure is determined by mass conservation, hydrostatic equilibrium and extended luminosity generation, while the radiative envelope structure is determined by the radiative transport and the central concentration of mass and luminosity.

Because the energy generation is principally proportional to the *ppI* nuclear cross section factor, $\varepsilon_o \sim S_{11}$, we obtain $T_c \sim S_{11}^{-0.134}$, in agreement with Iben (1969; 1991) and Castellani *et al.* (1993), who, however, incorrectly assumed $\rho \sim T^3$. This explains why in Figures 2, 4, and 6-8 of Hata and Langacker (1995), T_c-parametrization is equivalent to S_{11} parametrization, within the uncertainty of either. The *ppII*, *ppIII*, and *CNO* chains contributions to luminosity generation break this simple T_c form, adding a weak ρ_c dependence. The density exponents are:

$$\rho \sim \varepsilon_o^{-\psi} \kappa_o^{-\sigma} \mu^\xi T^\tau \quad , \quad \rho \sim \varepsilon_o^{-a} \kappa_o^{-b} \mu^c L_\odot^d \quad , \tag{16}$$

with

$$\psi = \sigma = \xi = \frac{1}{1 + \lambda + n}$$
$$\tau = \frac{3 - \nu + s}{1 + \lambda + n} \quad , \tag{17a}$$

$$a = \psi - (\epsilon\tau/\theta)$$
$$b = \sigma - (\zeta\tau/\theta)$$
$$c = \xi + (\eta\tau/\theta)$$
$$d = \tau/\theta \quad . \tag{17b}$$

The cases of the Kramers and OPAL opacities are presented in Table 2. The Böhm-Vitense opacity, with $\nu = 4$, is also shown for comparison.

A rotating core would change the hydrostatic equilibrium by adding the centrifugal force to that of gravity in the rest frame of solar matter. If, as in the magnetic case, the reverse influence of the thermomechanical structure on the rotation is neglected, the correction to the homology is:

$$T^4/P \sim \mu^4 m^2 [1 - \omega]^3$$
$$T^3/\rho \sim \mu^3 m^2 [1 - \omega]^3 \quad ,$$
$$\ell = \ell_o(\mu, T, \kappa_o, \varepsilon_o)[1 - \omega]^\chi \tag{18}$$
$$\chi = \frac{3(1 + \lambda)(3 + 2n)}{2(1 + \lambda + n)} - 3 \quad ,$$

where ℓ_o is the non-rotating luminosity function, $\omega \equiv \Omega^2 r^3/Gm$, and Ω is the angular rotation frequency. Near the center, $\omega \to 3\Omega_c^2/4\pi G\rho_c$. For typical SSMs, $\omega_c \approx 2\times10^{-7}$, using a reasonable solar core rotation rate (Elsworth 1995). The exponent χ is given in Table 2 for the three opacities. The rotation correction would be significant only for rotation rates ~ 400 times those in the Sun. Such a high rotation rate would not only change the thermomechanical structure, however, but could also induce sufficient chemical mixing to invalidate the homology.

An alternative to this homology is to treat the luminosity ℓ, not the radius r or the cumulative mass m, as the independent variable. Since the luminosity is a monotonically increasing function of r in the core, but not outside, this change of variables is feasible only in the core and separates out the luminosity-producing regions.

4. Core Homology and Neutrino Fluxes

After nuclear cross sections are introduced, and the ^3He, ^7Be abundances are assumed to be in steady state, each of the neutrino emissivities, $f_\nu(i)$, is a function of $\mathbf{X}$, ρ, T. Homology would then make each neutrino flux $\phi(i) \sim f_\nu(i)$, subject to the luminosity constraint (1). If there were only one power-law energy generation term in equation (1), the core homology would be exact and ρ_c, like all other core variables, would be a power of T_c only. The Be, B, and CNO neutrino production breakss this homology, so that, besides the principal sensitivity to T_c, the neutrino fluxes have a mild separate dependence on ρ_c. Using the luminosity constraint, Gough (1994) has obtained:

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\begin{aligned}
\phi(pp) &\sim \rho_c^{-0.1} \cdot T_c^{-0.7} \\
\phi(Be) &\sim \rho_c^{0.7} \cdot T_c^9 \sim \rho_c^{0.57} \cdot \phi(B)^{0.43} \\
\phi(B) &\sim \rho_c^{0.3} \cdot T_c^{21} \sim \rho_c^{-1.33} \cdot \phi(Be)^{2.33} \quad .
\end{aligned}
\tag{19}
$$

If we approximate $\rho \sim T^3$ in the solar core, we obtain $\phi(i) \sim T_C^{\alpha(i)}$, with $\alpha(pp) = -1$, $\alpha(Be) = 11$, $\alpha(B) = 22$; while Castellani *et al.* (1993), assuming an $n_{\text{eff}} = 3$ polytropic Sun, obtained $\alpha(pp) = -1.1$, $\alpha(Be) = 11$, $\alpha(B) = 27$. The small departure from core homology, together with uncertainties in the nuclear cross-section factors S_{34}, S_{17}, explains the scatter in diagrams plotting neutrino fluxes against T_c alone (Hata and Langacker 1994; 1995).

5. Core Homology and Helioseismology

Helioseismology, the study of sunquakes, is based on three distinct types of waves in the solar medium, p-modes, f-modes, and g-modes (Hansen and Kawaler 1994). The first two are acoustic, with pressure contrast as the restoring force, and are seen in the outer, convective zone, where they have much or most of their amplitudes. Their eigenfrequencies rise with the number of nodes in the successive modes.

The g-mode restoring force is gravity, and these modes have their largest amplitude in the core. Their eigenfrequencies *decrease* with the number of nodes. The g-modes have not yet been firmly detected by optical means, although they have perhaps been detected through their modulation of the solar wind (Thomson *et al.* 1995). All modes are labelled by eigennumbers n and l, with an azimuthal m if rotation is present. (Otherwise, the

eigenfrequencies are degenerate in m.) The ranges are: $n = 1, 2, ...,$ and $l = 0, 1, 2, ...$
For large n, the frequencies of the g-modes are given by (Hansen and Kawaler 1994):

$$\nu_g = \frac{\sqrt{l(l+1)}}{2n\pi^2} \, \Omega_g \quad ,$$

$$\Omega_g = \int_0^{R_\odot} dr \, \frac{N(r)}{r} \quad , \tag{20a}$$

$$N(r) = \sqrt{g(r)/\lambda_g(r)} \quad ,$$

where $g(r)$ is the local acceleration of gravity and $\lambda_g(r)$ a local scale height:

$$\frac{1}{\lambda_g} = \frac{1}{\lambda_\rho} - \frac{1}{\Gamma_{\mathrm{ad}}} \cdot \frac{1}{\lambda_P} = \frac{1}{\lambda_\rho}\left(1 - \frac{\Gamma}{\Gamma_{\mathrm{ad}}}\right) \quad , \tag{20b}$$

where $\Gamma_{\mathrm{ad}} = (d\ln P/d\ln\rho)_{\mathrm{ad}}$ is the adiabatic polytropic exponent.

Solar core homology can be applied to the integral Ω_g, which receives its main contribution from the core region. As $r \to 0$.

$$\Omega_g \simeq \frac{4\pi G}{3}\sqrt{\frac{\rho_c}{P_c}\left(\frac{1}{\Gamma} - \frac{1}{\Gamma_{\mathrm{ad}}}\right)_c} \cdot R_c \cdot \langle\bar\rho(R_c)\rangle \quad , \tag{21}$$

where R_c is the core radius ($R_c = (0.26)R_\odot$) and

$$\langle\bar\rho(R_c)\rangle = \frac{1}{R_c}\int_0^{R_c} dr \, \frac{3}{4\pi} \cdot \frac{m(r)}{r^3} \tag{22}$$

is the radially averaged mean density interior to R_c. Note $\Gamma_{\mathrm{ad}} \geq \Gamma$ implies convective stability of the core.

Because, outside the immediate central region ($x < 0.049$), the mass $m(r)$ rises more slowly than r^3, the integral emphasizes the central core as the dominant "yolk in the egg" mass concentration that controls the g-mode oscillations. A simple estimate is $\langle\bar\rho(R_c)\rangle \approx \rho_c$, but a better estimate results from applying core homology via the $n_{\mathrm{eff}} = \infty$ "isothermal" polytropic solution (Chandrasekhar 1939; Kippenhahn and Weigert 1990). The important length scale here is $\sqrt{P_c/4\pi G\rho_c^2} = 0.049R_\odot$. Using the power series and asymptotic properties of the solution, one obtains:

$$\langle\bar\rho(R_c)\rangle = \frac{3\rho_c}{z_c}\int_0^{z_c} dz \, \frac{dw(z)/dz}{z}$$
$$\simeq 0.56\rho_c \quad , \tag{23}$$

where $z_c = 0.26/0.049 = 5.3$ and $w(z)$ is the dimensionless gravitational potential. This estimate is smaller and more accurate than ρ_c as it covers the entire core, whose average density is lower than its central density.

As helioseismological observations are so far in good agreement with SSM predictions, we conclude that the homology presented in this paper is, for practical purposes, a complete parametrization for any "reasonable" changes to the minimal SSM.

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6. Conclusion

Assuming mechanical and thermal stasis and neglecting chemical evolution, the homology makes the (T_c, ρ_c) parametrization a general framework characterizing the inner core. Using this approach, we have disposed of three misconceptions: (1) that the luminosity-generating core of the Sun is polytropic; (2) that the polytropic relation $\rho \sim T^3$ is essential to understanding the Sun's core; and (3) that homology is inapplicable to stars on the middle of the main sequence. While the T-gradient depends on the opacity, T_c depends mainly on the mean molecular weight μ because of the homology, assuming a quiescent non-convective and unmixed core. The properties of the core depend only on one surface boundary condition, the total luminosity $L_\odot$, assumed to be in steady state with the core's luminosity. The surface temperature T_{eff} is then irrelevant.

The solar core is almost homologous because its luminosity generation is dominated by the ppI cycle and, over its narrow range of temperature and density, the opacity and luminosity generation can be approximated by power laws. The luminosity of solar models based on purely radiative transfer scales by the nuclear cross-section factor S_{11}, or equivalently, by $T_c \sim S_{11}^{-0.14}$. This quasi-homology justifies the T_c-parametrization as reasonable for estimating astrophysical uncertainties in any SSM and for extrapolating from any SSM to even extreme non-standard "cool Sun" or "hot Sun" models. In particular, analyses such as those of Hata and Langacker (1994; 1995) and of Bludman, Hata, Kennedy, and Langacker (1993), using the T_c-parametrization and nuclear cross section uncertainties alone, can arrive at theoretical neutrino flux and detection rate uncertainties and their correlations, agreeing with those Bahcall and Ulrich obtained from 1000 different Monte Carlo SSM simulations (Bahcall and Ulrich 1988).

Because of the central concentration of mass and luminosity generation, the Sun's outer radiative zone is nearly an $n_{\text{eff}} = 4.3$ partial polytrope, with exponential pressure, density, and temperature profiles. The density scale height $\lambda_\rho = 0.095 R_\odot$ is the single solar parameter entering into the MSW adiabaticity parameter that determines any small-angle MSW oscillations.
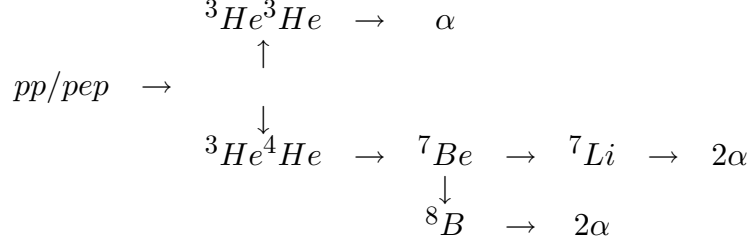
Appendix: The Luminosity Constraint

The constraint of the total photon luminosity $L_\odot$ upon the neutrino fluxes is almost independent of the specific SSM, resulting for the most part from the microscopic properties of the nuclear fusion reactions (Schwarzschild 1958; Cox and Giuli 1968; Bahcall and Ulrich 1988; Turck-Chièze 1988). The branching ratios assumed are mildly SSM-dependent and

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are taken from the results of Bahcall and Pinsonneault (1992), with helium diffusion. This luminosity constraint on the neutrino fluxes should be distinguished from that used to determine the helium abundance Y.

The Sun shines by two nuclear reaction chains, the dominant pp and the minor CNO. The pp chain itself consists of three subchains, ppI, $ppII$, and $ppIII$, each terminating in ^4He or α production in a different way.

$$
\begin{array}{ccccccccc}
& & ^3He^3He & \rightarrow & \alpha & & & & \\
& & \uparrow & & & & & & \\
pp/pep & \rightarrow & & & & & & & \\
& & \downarrow & & & & & & \\
& & ^3He^4He & \rightarrow & ^7Be & \rightarrow & ^7Li & \rightarrow & 2\alpha \\
& & & & \downarrow & & & & \\
& & & & ^8B & \rightarrow & 2\alpha & &
\end{array}
$$

For each reaction, the energy released, Q, is partitioned between neutrino energy Q_ν and photon luminosity Q_γ, so that $Q = Q_\gamma + Q_\nu$. Reactions without neutrinos have $Q_\nu = 0$.

The luminosity constraint arises from the proportionality of the neutrinos fluxes to the nuclear reaction rates. Each reaction contributes its Q_γ value to the photon luminosity. Since there are *two* neutrinos emitted for each reaction chain, the luminosity is related to the neutrino fluxes by

$$
L_\odot = \left(\sum_i Q_\gamma(i)\phi(i) \right) \cdot (4\pi R_\odot^2/2) \quad , \tag{A.1}
$$

summed over all nuclear reactions i. Normalizing to the ppI Q_γ value and to the neutrino fluxes measured at the Earth's orbit,

$$
\frac{\sum_i Q_\gamma(i)\phi(i)}{Q_\gamma(ppI)} = \frac{2}{4\pi R_\odot^2} \cdot \frac{L_\odot}{Q_\gamma(ppI)} \cdot \left(\frac{R_\odot}{r_\oplus}\right)^2 \quad , \tag{A.2}
$$

where $r_\oplus = 149.6 \times 10^6$ km is the Earth's average orbital radius.

The Q, Q_ν, and Q_γ values for all four reaction chains are listed in Table 3 (Fowler 1967; Bahcall and Ulrich 1988; Turck-Chièze *et al.* 1988). The pp chain is initiated by either the pp or pep reactions, in the ratio $99.6\% : 0.4\%$; the $ppII$ chain emits neutrinos at two discrete energies, in the ratio $89.7\% : 10.3\%$. The quoted energies are weighted averages. The ratios of the Q_γ values can then be computed to obtain the constraint (1).

Also necessary are the ratios of the different neutrino fluxes in order to obtain equation (2,3). These fluxes and the percentages above are obtained from a specific SSM, but the coefficients in the constraints (1-3) are relatively insensitive to theoretical variations that do not depart radically from conventional SSMs. The ratios of the reaction subchain termination rates are related to the fluxes by:

$$
\begin{aligned}
\frac{\text{term}(ppII)}{\text{term}(ppI)} &= \frac{\phi(Be)}{[\phi(pp)/2] - \phi(Be)} \quad , \\
\frac{\text{term}(ppIII)}{\text{term}(ppII)} &= \frac{\phi(B)}{\phi(Be)} \quad .
\end{aligned} \tag{A.3}
$$

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TABLE 1

Luminosity Power Laws[a]

Opacity	n	s	δ	ϵ	ζ	η	θ
Kramers	1	3.5	5.44	0.833	0.167	1.33	6.12
BV	0.5	2.5	4.83	0.8	0.2	1.3	5.8
OPAL	0.43	2.47	4.81	0.794	0.206	1.29	5.99

[a]*Exponents in the luminosity power-laws (15) for three Rosseland mean opacities of the form (7).*

TABLE 2

Density Power Laws[a]

Opacity	ψ	τ	a	b	c	d	χ
Kramers	0.333	0.753	0.231	0.313	0.497	0.123	2
BV	0.4	0.6	0.317	0.372	0.534	0.103	1.8
OPAL	0.412	0.507	0.345	0.395	0.521	0.0847	1.77

[a]*Exponents in the density power-laws (16) for three Rosseland mean opacities of the form (7). See text for last entry (χ).*

TABLE 3

pp Reaction Q Values[a]

Chain	Q	Q_ν	Q_γ
$ppI(pp)$	26.732	0.540	26.192
$ppII(Be)$	26.682	1.083	25.599
$ppIII(B)$	26.639	6.980	19.659

[a]*All energies in MeV. Q_ν is spectrum-averaged for each reaction chain.*

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Figure Captions

Fig. 1. The stiffness coefficient $d\ln P/d\ln\rho$ across the Sun's profile. In the convective zone, $\Gamma = 5/3$; in the outer radiative zone, $\Gamma \approx 1.23$; but over the core, $\Gamma \sim 1$ and varies.

Fig. 2. The temperature gradient $\nabla = d\ln T/d\ln P$ and $1 - \Gamma^{-1} = d\ln(P/\rho)/d\ln P$. Where the chemical composition is homogeneous, $\nabla = 1 - \Gamma^{-1} = 0.4$ in the convective zone and ≈ 0.18 in the outer radiative zone. But over the core, $P/(\Re\rho T) = (1 + D)/\mu$ is varying.

Fig. 3. The lines shows calculated L–M and R–M relations for a large range of zero-age Main Sequence stars. The dots and triangles show best measurements of slected Main Sequence components of detached and visual binary components, respectively. (From Kippenhahn and Weigert (1990), by permission.)